[11-01-27-T12] REV

Mean Value Theorem

ROLLE'S THEOREM. Let f be a function which is continuous everywhere on the closed interval $[a, b]$ and has a derivative at each point of the open interval (a, b). Further suppose that $f(a) = f(b)$. *Then there is at least one point c in the open interval* (a, b) *such that* $f'(c) = 0$.

Proof. The existence of both a maximum and a minimum value of f is guaranteed by the max-min existence theorem, because f is continuous on $[a, b]$. If both the maximum and the minimum values of f occur at the endpoints of the interval, then we are done, because $f'(x) = 0$ for every x in (a, b). So, suppose that the maximum and the minimum values of f do not both occur at the endpoints. By hypothesis f is differentiable at every point in (a, b), so by the critical point theorem, there is a point in (a, b) at which the value of the derivative of f is zero. That is, there is a number $c \in (a, b)$ such that $f'(c) = 0$. □

MEAN-VALUE THEOREM. Let f be a function which is continuous everywhere on the closed interval $[a, b]$ and has a derivative at each point of the open interval (a, b). *Then there is at least one interior point c of* (a, b) *for which*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Define a function F on $[a, b]$ by

$$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

Then,

$$F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

Note that F is continuous on $[a, b]$ and differentiable on (a, b) because by hypothesis f possess these two qualities and certainly $f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$ does, too. Furthermore, $F(a) = F(b)$. By Rolle's Theorem, there exists a number $c \in (a, b)$ such that $F'(c) = 0$. Therefore, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

□

CAUCHY'S MEAN-VALUE THEOREM. Let f and g be both continuous everywhere on the closed interval $[a, b]$ and differentiable at each point of the open interval (a, b). If $g'(x) \neq 0$ for all $x \in (a, b)$, then $g(a) \neq g(b)$ and there exists a number $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Well, the proof is left to you.

Hint: define F on $[a, b]$ by $F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)} = (g(x) - g(a))$.

■ Assignment.

Your textbook (Varberg) at page 200 provides a proof of the monotonicity theorem and of a simple but important theorem that states that if the derivatives of two functions are equal, the functions differ by at most a constant. You should work through both of the page 200 proofs and be able to produce the proofs on an exam.

■ Optional.

Prove the following generalization of Cauchy's Mean Value Theorem that guarantees the existence of a number $\xi \in (a, b)$ such that

$$\frac{f(b)-f(a)-\sum_{k=1}^{n-1} \frac{(b-a)^k}{k!}\, f^{(k)}(a)}{g(b)-g(a)-\sum_{k=1}^{n-1} \frac{(b-a)^k}{k!}\, g^{(k)}(a)} = \frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}.$$